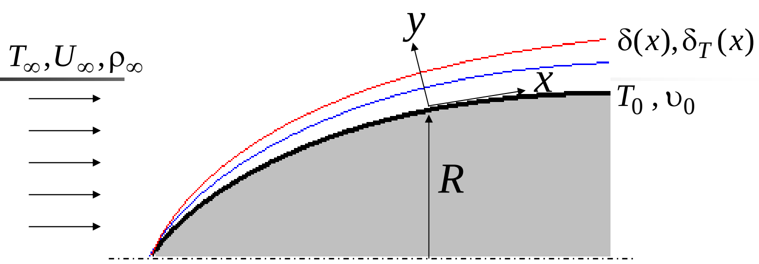
## Boundary Layer Integration Analysis




Assumptions: incompressible, no external sources, no swirling

Considerations: $U_\infty(x)$ (pressure gradients)

$\rho_\infty(x)$ (stratification)

$\upsilon_0(x)$ (blowing/suction)

$T_0(x)$ (nonuniform wall temperature)

$R(x)$ (change of body shape)

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## Boundary Layer Integration Analysis

**Boundary Layer Approximations:**

- x-momentum ($u$) >> y-momentum ($v$-$v_0$)

  (x-convection >> y-convection)

- y-derivatives >> x-derivatives: $\frac{\partial u}{\partial y} >> \frac{\partial u}{\partial x}$ and $\frac{\partial T}{\partial y} >> \frac{\partial T}{\partial x}$

  (y-diffusion >> x-diffusion)

- pressure x-derivative >> pressure y-derivative: $\frac{\partial p}{\partial x} \approx \frac{dp}{dx}$

$$\left(\frac{1}{\rho}\frac{dp}{dx}\right)_{\text{within b.l.}} \approx \left(\frac{1}{\rho}\frac{dp}{dx}\right)_{\text{out of b.l.}} = -U_\infty \frac{dU_\infty}{dx} \quad \text{(Bernoulli equation)}$$

- no swirling

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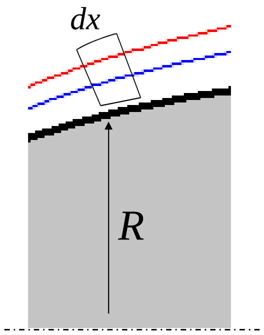
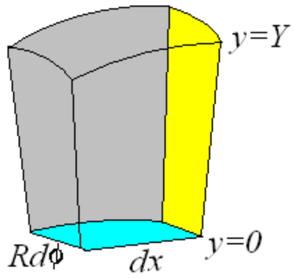
## Boundary Layer Integration Analysis

### Control Volume




$d\forall = Rd\phi \cdot \delta x \cdot Y$

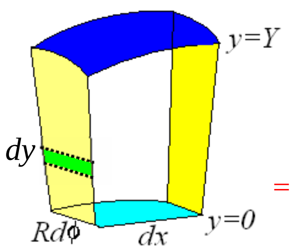


## Boundary Layer Integration Analysis

### Mass Conservation

Assumption: no swirling, $R >> Y > \delta, \delta_T$

mass inflow rate = mass outflow rate




$d\forall = Rd\phi \cdot \delta x \cdot Y$

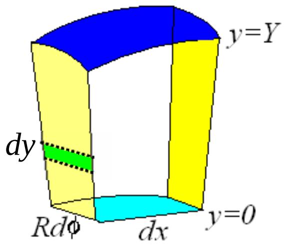
$$\int_0^Y \rho u R d\phi dy + \rho_0 \upsilon_0 R d\phi dx$$

$$= \left\{\int_0^Y \rho u R d\phi dy\right\} + \frac{d}{dx}\left\{\int_0^Y \rho u R d\phi dy\right\} dx + \rho_Y \upsilon_Y (R + Y) d\phi dx$$

$$\rho_Y \upsilon_Y = \rho_0 \upsilon_0 - \frac{1}{R}\frac{d}{dx}\left\{R\int_0^Y \rho u dy\right\}$$

## Boundary Layer Integration Analysis

### Momentum Conservation




net x-force = net x-momentum outflow rate

momentum inflow rate = $\int_0^Y \rho u^2 R d\phi dy$

$+ 0 \cdot \rho_0 \upsilon_0 R d\phi dx$

momentum outflow rate =

$$\left\{\int_0^Y \rho u^2 R d\phi dy\right\} + \frac{d}{dx}\left\{\int_0^Y \rho u^2 R d\phi dy\right\} dx$$

$$+\ u_Y \cdot \rho_Y \upsilon_Y (R + Y) d\phi dx$$

for $R >> Y$

net momentum outflow rate =

$$\frac{d}{dx}\left\{\int_0^Y \rho u^2 R d\phi dy\right\} dx + u_Y \cdot \rho_Y \upsilon_Y R d\phi dx$$

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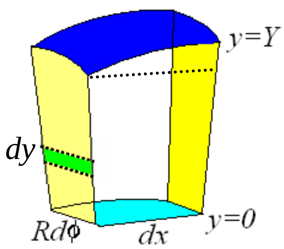
## Boundary Layer Integration Analysis

### Momentum Conservation




pressures forces = $\left(\int_0^Y pR d\phi dy\right)$

$$-\left\{\left(\int_0^Y pR d\phi dy\right) + \frac{d}{dx}\left(\int_0^Y pR d\phi dy\right) dx\right\}$$

$$+ p \frac{d}{dx}(R d\phi Y) dx$$

shear stresses = $-\tau_0 R d\phi dx + \tau_Y (R + Y) d\phi dx$

for $Y > \delta$

net force $= -\dfrac{d}{dx}\left(\displaystyle\int_0^Y pR d\phi dy\right) dx + p \dfrac{d}{dx}(R d\phi Y) dx - \tau_0 R d\phi dx$

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## Boundary Layer Integration Analysis

### Momentum Conservation

$$\text{net force} = -\frac{d}{dx}\left(\int_0^Y pRd\phi dy\right)dx + p\frac{d}{dx}(Rd\phi Y)dx - \tau_0 Rd\phi dx$$

$$= \text{net momentum outflow rate}$$

$$= \frac{d}{dx}\left\{\int_0^Y \rho u^2 Rd\phi dy\right\}dx + u_Y \cdot \rho_Y \upsilon_Y Rd\phi dx$$

Because: $p = p(x)$ and $u_Y = U_\infty$ and $\rho_Y \upsilon_Y = \rho_0 \upsilon_0 - \frac{1}{R}\frac{d}{dx}\left\{R\int_0^Y \rho u dy\right\}$

$$-\tau_0 = \rho_0\upsilon_0 U_\infty - Y\rho_\infty U_\infty \frac{dU_\infty}{dx} - \frac{U_\infty}{R}\frac{d}{dx}\left(R\int_0^Y \rho u dy\right) + \frac{1}{R}\frac{d}{dx}\left(R\int_0^Y \rho u^2 dy\right)$$

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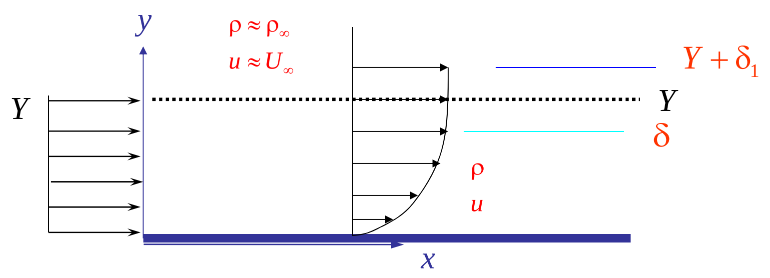
## Boundary Layer Integration Analysis

displacement thickness $\delta_1 \equiv \int_0^\infty \left(1 - \frac{\rho u}{\rho_\infty U_\infty}\right)dy \approx \int_0^Y \left(1 - \frac{\rho u}{\rho_\infty U_\infty}\right)dy$




for $Y > \delta$: $\int_0^Y \rho_\infty U_\infty dy = \int_0^{Y+\delta_1} \rho u dy = \int_0^Y \rho u dy + \rho_\infty U_\infty \delta_1$

$$\rho_\infty U_\infty \delta_1 = \int_0^Y (\rho_\infty U_\infty - \rho u)dy$$

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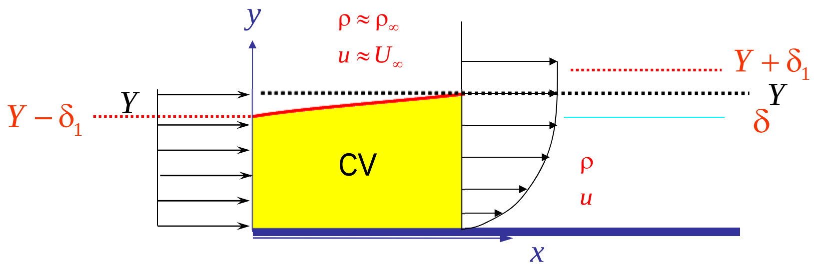
## Boundary Layer Integration Analysis

momentum thickness $\delta_2 \equiv \int_0^\infty \frac{\rho u}{\rho_\infty U_\infty}\left(1-\frac{u}{U_\infty}\right)dy \approx \int_0^Y \frac{\rho u}{\rho_\infty U_\infty}\left(1-\frac{u}{U_\infty}\right)dy$




$$\dot{m} = \int_0^Y \rho u\, dy = \rho_\infty U_\infty (Y - \delta_1)$$

$$momentum\ \ loss = \dot{m}\cdot U_\infty - \int_0^Y \rho u^2 dy = \int_0^Y U_\infty \cdot \rho u dy - \int_0^Y \rho u^2 dy \ \ \equiv \rho_\infty U_\infty^2 \delta_2 \ \ = drag$$

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## Boundary Layer Integration Analysis

$$-\tau_0 = \rho_0 \upsilon_0 U_\infty - Y\rho_\infty U_\infty \frac{dU_\infty}{dx} - \frac{U_\infty}{R}\frac{d}{dx}\left(R\int_0^Y \rho u\, dy\right) + \frac{1}{R}\frac{d}{dx}\left(R\int_0^Y \rho u^2 dy\right)$$

⇩

$$\frac{\tau_0}{\rho_\infty U_\infty^2} = \frac{d\delta_2}{dx} - \frac{\rho_0 \upsilon_0}{\rho_\infty U_\infty} + \delta_2\left\{\left(2+\frac{\delta_1}{\delta_2}\right)\frac{1}{U_\infty}\frac{dU_\infty}{dx} + \frac{1}{\rho_\infty}\frac{d\rho_\infty}{dx} + \frac{1}{R}\frac{dR}{dx}\right\}$$

blowing/suction

pressure gradients

stratification

change of body shape

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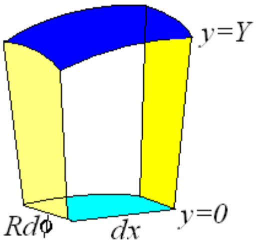
## *Boundary Layer Integration Analysis*

### Energy Conservation

net energy outflow rate =
heat sources + work rates done on CV




| |
|---|
| Assumptions: no external sources |
| negligible pressure work |

work rate by shear forces = $\dot{W}(0)+\dot{W}(Y)$

$$= -\tau_0 R d\phi dx \cdot \cancel{u_0} \;+\; \cancel{\tau_Y} R d\phi dx \cdot u_Y = 0$$

net energy outflow rate = 0

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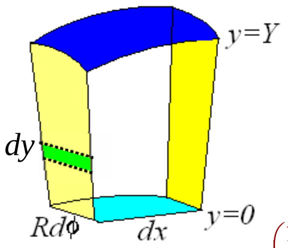
## *Boundary Layer Integration Analysis*

### Energy Conservation

total energy $i$ = internal + kinetic + pressure energy

**Assume: negligible x-diffusion**




energy inflow rate =

$$\cancel{\int_0^Y i\cdot \rho u R d\phi dy} + i_0 \cdot \rho_0 \upsilon_0 R d\phi dx + q_0'' \cdot R d\phi dx$$

energy outflow rate =

$$\cancel{\left(\int_0^Y i\cdot \rho u R d\phi dy\right)} + \frac{d}{dx}\left(\int_0^Y i\cdot \rho u R d\phi dy\right)dx + i_Y \cdot \rho_Y \upsilon_Y R d\phi dx + \cancel{q_Y''} \cdot R d\phi dx$$

$$\Rightarrow \frac{d}{dx}\left(\int_0^Y i\cdot \rho u R dy\right) + i_Y \cdot \rho_Y \upsilon_Y R - i_0 \cdot \rho_0 \upsilon_0 R - q_0'' \cdot R = 0$$

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## *Boundary Layer Integration Analysis*

### Energy Conservation

total energy $i$ = internal + kinetic + pressure energy

$$\frac{d}{dx}\left(\int_0^Y i\cdot\rho u R dy\right) + i_Y \cdot \rho_Y \upsilon_Y R - i_0 \cdot \rho_0 \upsilon_0 R - q_0'' \cdot R = 0$$

$i_Y \approx i_\infty$

$$\rho_Y \upsilon_Y = \rho_0 \upsilon_0 - \frac{1}{R}\frac{d}{dx}\left\{R\int_0^Y \rho u dy\right\}$$

$$\frac{d}{dx}\left(R\int_0^Y i\cdot\rho u dy\right) + i_\infty \cdot R\left\{\rho_0 \upsilon_0 - \frac{1}{R}\frac{d}{dx}\left\{R\int_0^Y \rho u dy\right\}\right\} - i_0 \cdot \rho_0 \upsilon_0 R - q_0'' \cdot R = 0$$

$$q_0'' = \frac{1}{R}\frac{d}{dx}\left(R\int_0^Y \rho u(i - i_\infty) dy\right) - \rho_0 \upsilon_0 (i_0 - i_\infty)$$



## *Boundary Layer Integration Analysis*

### Energy Conservation

$$q_0'' = \frac{1}{R}\frac{d}{dx}\left(R\int_0^Y \rho u(i - i_\infty) dy\right) - \rho_0 \upsilon_0 (i_0 - i_\infty)$$

$$i = e + \tfrac{1}{2}|\vec{u}|^2 + \tfrac{p}{\rho} \approx e + \tfrac{1}{2}u^2 + \tfrac{p}{\rho} = h + \tfrac{1}{2}u^2 \approx h \qquad \because\ Ec << 1$$

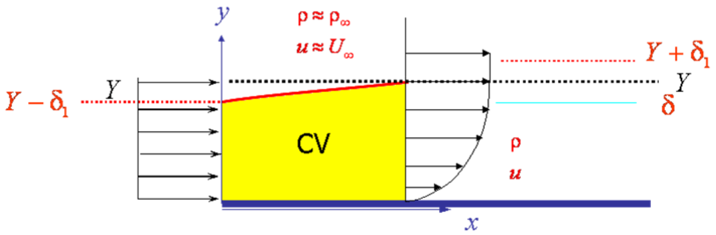
$$h = c_p T$$

if constant $c_p$ ⇨ $$q_0'' = \frac{1}{R}\frac{d}{dx}\left(R\int_0^Y \rho u c_p (T - T_\infty) dy\right) - \rho_0 \upsilon_0 c_p (T_0 - T_\infty)$$

## *Boundary Layer Integration Analysis*

### Thermal Boundary Layer Thickness

enthalpy thickness $\Delta_2 \equiv \int_0^\infty \rho u(i-i_\infty)\,dy \Big/ \rho_\infty U_\infty (i_0 - i_\infty)$




$$\dot{m} = \int_0^Y \rho u dy = \rho_\infty U_\infty (Y - \delta_1)$$

energy gain $= \int_0^Y i \cdot \rho u dy - i_\infty \cdot \dot{m} = \int_0^Y i \cdot \rho u dy - i_\infty \cdot \int_0^Y \rho u dy \;\equiv\; (i_0 - i_\infty)\rho_\infty U_\infty \Delta_2$

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## *Boundary Layer Integration Analysis*

### Thermal Boundary Layer Thickness

enthalpy thickness
$$\Delta_2 \equiv \frac{\int_0^\infty \rho u(i-i_\infty)\,dy}{\rho_\infty U_\infty (i_0 - i_\infty)}$$

$$\approx \frac{\int_0^\infty \rho u\left(c_p T - c_{p\infty} T_\infty\right) dy}{\rho_\infty U_\infty \left(c_{p0} T_0 - c_{p\infty} T_\infty\right)} \qquad \because \;\; \mathrm{Ec} << 1$$

$$\approx \frac{\int_0^\infty \rho u(T - T_\infty)\,dy}{\rho_\infty U_\infty (T_0 - T_\infty)} \qquad \text{if constant } c_p$$

$$\approx \frac{\int_0^\infty u(T - T_\infty)\,dy}{U_\infty (T_0 - T_\infty)} \qquad \text{for } \rho = \rho(x) = \rho_\infty(x)$$

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## Boundary Layer Integration Analysis

### Energy Conservation

$$\Delta_2 = \frac{\int_0^\infty u\left(T - T_\infty\right)dy}{U_\infty\left(T_0 - T_\infty\right)}$$

$$q_0'' = \frac{1}{R}\frac{d}{dx}\left(R\int_0^Y \rho_\infty u c_p\left(T - T_\infty\right)dy\right) - \rho_0 \upsilon_0 c_p\left(T_0 - T_\infty\right)$$

$$\frac{d}{dx}\left(R\int_0^Y \rho_\infty u c_p\left(T - T_\infty\right)dy\right) = \frac{d}{dx}\left(\rho_\infty c_p R\int_0^Y u\left(T - T_\infty\right)dy\right)$$

$$= c_p\frac{d}{dx}\left\{\rho_\infty R U_\infty\left(T_0 - T_\infty\right)\Delta_2\right\}$$

$$\frac{1}{\rho_\infty U_\infty c_p\left(T_0 - T_\infty\right)}\frac{d}{dx}\left(R\int_0^Y \rho_\infty u c_p\left(T - T_\infty\right)dy\right)$$

$$= \frac{1}{\rho_\infty U_\infty\left(T_0 - T_\infty\right)}\frac{d}{dx}\left\{\rho_\infty R U_\infty\left(T_0 - T_\infty\right)\Delta_2\right\}$$

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## Boundary Layer Integration Analysis

### Energy Conservation

$$\frac{q_0''}{\rho_\infty c_p U_\infty\left(T_0 - T_\infty\right)} = \frac{1}{\rho_\infty U_\infty\left(T_0 - T_\infty\right)}\frac{1}{R}\frac{d}{dx}\left\{\rho_\infty R U_\infty\left(T_0 - T_\infty\right)\Delta_2\right\} - \frac{\rho_0\upsilon_0}{\rho_\infty U_\infty}$$

$$\frac{q_0''}{\rho_\infty U_\infty c_p\left(T_0 - T_\infty\right)} = \frac{d\Delta_2}{dx} - \frac{\rho_0\upsilon_0}{\rho_\infty U_\infty}$$

wall temperature variations

$$+\Delta_2\left\{\frac{1}{U_\infty}\frac{dU_\infty}{dx} + \frac{1}{\rho_\infty}\frac{d\rho_\infty}{dx} + \frac{1}{R}\frac{dR}{dx} + \underline{\frac{1}{\left(T_0 - T_\infty\right)}\frac{d\left(T_0 - T_\infty\right)}{dx}}\right\}$$

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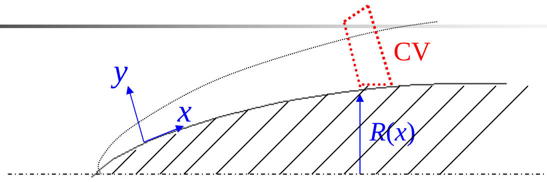
## Boundary Layer Integration Analysis




$$\frac{\tau_0}{\rho_\infty U_\infty^2} = -\frac{\rho_0 \upsilon_0}{\rho_\infty U_\infty} + \frac{d\delta_2}{dx} + \delta_2 \left\{ \left(2 + \frac{\delta_1}{\delta_2}\right) \frac{1}{U_\infty}\frac{dU_\infty}{dx} + \frac{1}{\rho_\infty}\frac{d\rho_\infty}{dx} + \frac{1}{R}\frac{dR}{dx} \right\}$$

$$\frac{q_0''}{\rho_\infty U_\infty c_p \left(T_0 - T_\infty\right)} = -\frac{\rho_0 \upsilon_0}{\rho_\infty U_\infty} + \frac{d\Delta_2}{dx} + \Delta_2 \left\{ \frac{1}{U_\infty}\frac{dU_\infty}{dx} + \frac{1}{\rho_\infty}\frac{d\rho_\infty}{dx} + \frac{1}{R}\frac{dR}{dx} + \frac{1}{\left(T_0 - T_\infty\right)} \frac{d\left(T_0 - T_\infty\right)}{dx} \right\}$$

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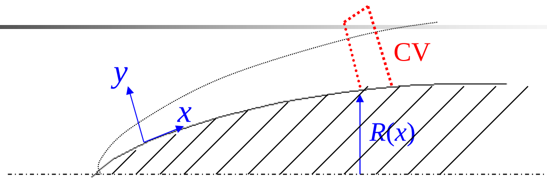
## Boundary Layer Integration Analysis




$$-\tau_0 = \rho_0 \upsilon_0 U_\infty - \rho_\infty U_\infty \frac{dU_\infty}{dx} - \frac{U_\infty}{R}\frac{d}{dx}\left( R\int_0^Y \rho u dy \right) + \frac{1}{R}\frac{d}{dx}\left( R\int_0^Y \rho u^2 dy \right)$$

$$q_0'' = \frac{1}{R}\frac{d}{dx}\left( R\int_0^Y \rho_\infty u c_p \left(T - T_\infty\right) dy \right) - \rho_0 \upsilon_0 c_p \left(T_0 - T_\infty\right)$$

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## Boundary Layer Integration Analysis

*uniform flow over a flat-plate*

flat-plate flow: $U_\infty = \text{constant}$ and $\frac{dR}{dx} = 0$

Assumptions:

- constant density: $\rho = \text{constant} = \rho_\infty$
- constant properties: $k, c_p, \mu$ etc.
- uniform wall temperature: $T_0 = \text{constant}$
- no blowing/suction: $\upsilon_0 = 0$
- kinetic energy << thermal energy: $Ec << 1$

$$\frac{\tau_0}{\rho_\infty U_\infty^2} = \frac{d\delta_2}{dx}$$

$$\frac{q_0''}{\rho_\infty U_\infty c_p (T_0 - T_\infty)} = \frac{d\Delta_2}{dx}$$



## *Uniform flow over a flat-plate*

*~ momentum boundary layer ~*

$$\frac{\tau_0}{\rho_\infty U_\infty^2} = \frac{d\delta_2}{dx}$$

■ guess velocity profile within the boundary layer

e.g. polynomial of 3rd order: $\frac{u}{U_\infty} = \frac{\eta}{2}\left(3 - \eta^2\right)$ where $\eta \equiv y/\delta(x)$

B.C.s: $u(\eta = 0) = 0$, $u(\eta = 1) = U_\infty$, $\frac{\partial u}{\partial y}(\eta = 1) = 0$, $\frac{\partial^2 u}{\partial y^2}(\eta = 0) = 0$

$$\Rightarrow \quad \delta_2 = \int_0^\infty \frac{\rho u}{\rho_\infty U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy = \int_0^\delta \frac{\rho u}{\rho_\infty U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy = \frac{39}{280}\delta \quad , \quad \tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{3\mu}{2\delta} U_\infty$$

$$\Rightarrow \quad 2\delta \frac{d\delta}{dx} = \frac{280}{13} \cdot \frac{\mu}{\rho_\infty U_\infty} \qquad I.C. \quad \delta(0) = 0$$

$$\Rightarrow \quad \delta(x) = \sqrt{\frac{280}{13}} \cdot x \cdot \mathrm{Re}_x^{-1/2} \approx 4.64 \cdot x \cdot \mathrm{Re}_x^{-1/2}$$

$$C_f = 3\sqrt{\frac{13}{280}} \mathrm{Re}_x^{-1/2} \approx 0.646 \, \mathrm{Re}_x^{-1/2}$$

## *Uniform flow over a flat-plate*

### *~ thermal boundary layer ~*

$$\frac{q_0''}{\rho_\infty U_\infty c_p (T_0 - T_\infty)} = \frac{d\Delta_2}{dx} = \frac{d}{dx}\left( \frac{\int_0^\infty \rho u (T - T_\infty) dy}{\rho_\infty U_\infty (T_0 - T_\infty)} \right) = \frac{d}{dx}\left( \frac{\int_0^{\delta_T} \rho u (T - T_\infty) dy}{\rho_\infty U_\infty (T_0 - T_\infty)} \right)$$

- guess temperature profile within the boundary layer

e.g. polynomial of 3rd order:

$$\Theta \equiv \frac{T_0 - T}{T_0 - T_\infty} = \frac{\xi}{2}\left(3 - \xi^2\right) \quad \text{where} \quad \xi \equiv y/\delta_T(x)$$

B.C.s: $\Theta(\xi = 0) = 0\,,\ \Theta(\xi = 1) = 1\,,\ \frac{\partial T}{\partial y}(\xi = 1) = 0\ ,\ \frac{\partial^2 T}{\partial y^2}(\xi = 0) = 0$

~ No net conduction at wall because there is no convection there.

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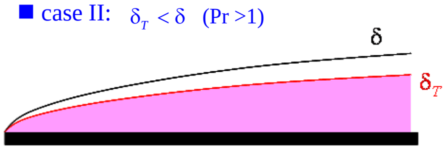
$$\Delta_2 = \frac{\int_0^{\delta_T} u(T - T_\infty) dy}{U_\infty (T_0 - T_\infty)}$$

$$\Delta \equiv \delta_T / \delta$$

- case II: $\delta_T < \delta$ (Pr >1)



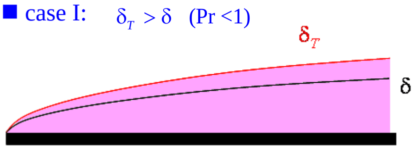

$$\Delta_2 = \left(\frac{3}{20}\Delta - \frac{3}{280}\Delta^3\right)\cdot \delta_T \equiv f(\Delta)\cdot\delta_T$$

for $0 \le y \le \delta_T$ :

$$\frac{u}{U_\infty} = \frac{\eta}{2}\left(3 - \eta^2\right)$$

$$\Theta \equiv \frac{T_0 - T}{T_0 - T_\infty} = \frac{\xi}{2}\left(3 - \xi^2\right)$$

- case I: $\delta_T > \delta$ (Pr <1)




$$\Delta_2 = \left(-1 + 14\Delta^2 - 35\Delta^3 + 35\Delta^4\right)\frac{3}{280\Delta^4}\cdot \delta_T \equiv f(\Delta)\cdot\delta_T$$

for $0 \le y \le \delta_T$ :

$$\frac{u}{U_\infty} = \begin{cases} \frac{\eta}{2}\left(3 - \eta^2\right) & \text{for } 0 \le y \le \delta \\ 1 & \text{for } \delta \le y \le \delta_T \end{cases}$$

$$\Theta \equiv \frac{T_0 - T}{T_0 - T_\infty} = \frac{\xi}{2}\left(3 - \xi^2\right)$$

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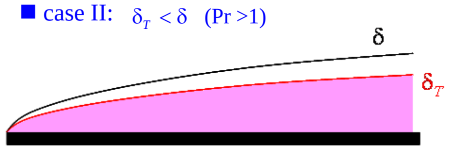
$$\Delta_2 = \frac{\int_0^{\delta_T} u(T - T_\infty)dy}{U_\infty (T_0 - T_\infty)}$$

$$\Delta \equiv \delta_T / \delta$$

$$\xi \equiv y/\delta_T$$

$$\eta \equiv y/\delta = y/\delta_T \cdot \delta_T/\delta = \xi \cdot \Delta$$

■ case II: $\delta_T < \delta \quad (\Pr > 1)$




$$\frac{u}{U_\infty} = \frac{\eta}{2}\left(3 - \eta^2\right)$$

$$\Theta \equiv \frac{T_0 - T}{T_0 - T_\infty} = \frac{\xi}{2}\left(3 - \xi^2\right)$$

$$(T - T_\infty) = (T - T_0) + (T_0 - T_\infty) = (T_0 - T_\infty)(-\Theta + 1)$$

$$\int_0^{\delta_T} u(T - T_\infty)dy = U_\infty (T_0 - T_\infty)\int_0^{\delta_T} \frac{\eta}{2}\left(3 - \eta^2\right)\cdot\left[1 - \frac{\xi}{2}\left(3 - \xi^2\right)\right]dy$$

$$= U_\infty (T_0 - T_\infty)\int_0^1 \Delta\frac{\xi}{2}\left(3 - \Delta^2\xi^2\right)\cdot\left[1 - \frac{\xi}{2}\left(3 - \xi^2\right)\right]\delta_T \, d\xi$$

$$\Delta_2 = \delta_T \cdot \Delta\int_0^1 \frac{\xi}{2}\left(3 - \Delta^2\xi^2\right)\left[1 - \frac{\xi}{2}\left(3 - \xi^2\right)\right]d\xi = \delta_T \cdot f(\Delta)$$

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## *Uniform flow over a flat-plate*

### *~ thermal boundary layer ~*

$$\Theta \equiv \frac{T_0 - T}{T_0 - T_\infty} = \frac{\xi}{2}\left(3 - \xi^2\right) \qquad \xi \equiv y/\delta_T \qquad \Delta \equiv \delta_T/\delta$$

$$\frac{q_0''}{\rho_\infty U_\infty c_p (T_0 - T_\infty)} = \frac{d\Delta_2}{dx}$$

$$\Delta_2 = f(\Delta)\delta_T$$

⇓

$$\delta_T \frac{d}{dx} f(\Delta)\delta_T = \frac{3}{2}\frac{\alpha}{U_\infty}$$

⇓

$$\delta_T \frac{d\delta_T}{dx} = \frac{3}{2 f(\Delta)}\frac{\alpha}{U_\infty}$$

$$\delta(x) = \sqrt{\frac{280}{13}\frac{\nu x}{U_\infty}}$$

■ wall heat flux:

$$q_0'' = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{3k}{2\delta_T}(T_0 - T_\infty)$$

■ Momentum and thermal boundary layers begin at the same location: *I.C.* $\delta(0) = 0$ and $\delta_T(0) = 0$

From the scaling analysis we know that

$$\Delta \equiv \frac{\delta_T}{\delta} = \Delta(\Pr) \neq \Delta(x)$$

$$\delta_T(x) = \left(\frac{3\alpha x}{U_\infty f(\Delta)}\right)^{1/2}$$

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## Uniform flow over a flat-plate
### ~ thermal boundary layer ~

$$\delta_T(x) = \left(\frac{3\alpha x}{U_\infty f(\Delta)}\right)^{1/2} \qquad \Delta \equiv \delta_T/\delta$$

$$\delta(x) = \sqrt{\frac{280}{13}\frac{\nu x}{U_\infty}}$$

$$\delta_T^2 = \frac{3\alpha x}{U_\infty f(\Delta)} = \frac{280}{13}\frac{\nu x}{U_\infty}\cdot\frac{13}{280}\cdot\frac{3\alpha}{\nu f(\Delta)} = \delta^2\cdot\frac{39}{280\,\mathrm{Pr}\cdot f(\Delta)}$$

$$\Delta^2 f(\Delta) = \frac{39}{280}\mathrm{Pr}^{-1}$$

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## Uniform flow over a flat-plate
### ~ thermal boundary layer ~

$$\Delta^2 f(\Delta) = \frac{39}{280}\mathrm{Pr}^{-1}$$

- case II: $\delta_T << \delta \quad (\mathrm{Pr} >> 1) \quad$ i.e. $\Delta = \delta_T/\delta << 1$

$$f(\Delta) = \left(\frac{3}{20}\Delta - \frac{3}{280}\Delta^3\right) \approx \frac{3}{20}\Delta \quad \Rightarrow \quad \Delta^2\cdot\frac{3\Delta}{20} = \frac{39}{280}\mathrm{Pr}^{-1} \Rightarrow \Delta = \left(\frac{13}{14}\right)^{1/3}\mathrm{Pr}^{-1/3}$$

$$\delta(x) = \sqrt{\frac{280}{13}\frac{\nu x}{U_\infty}} = \sqrt{\frac{280}{13}}\cdot x\cdot \mathrm{Re}_x^{-1/2}$$

$$\delta_T(x) = \Delta\cdot\delta(x) = \left(\frac{13}{14}\right)^{1/3}\mathrm{Pr}^{-1/3}\cdot\sqrt{\frac{280}{13}}\cdot x\cdot\mathrm{Re}_x^{-1/2} = 4.53\cdot x\cdot\mathrm{Re}_x^{-1/2}\cdot\mathrm{Pr}^{-1/3}$$

- wall heat flux: $q_0'' = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{3k}{2\delta_T}(T_0 - T_\infty)$

$$Nu = \frac{hx}{k} = \frac{q_0'' x}{k(T_0 - T_\infty)} = \frac{3x}{2\delta_T} = 0.331\mathrm{Re}_x^{1/2}\cdot\mathrm{Pr}^{1/3}$$

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## Uniform flow over a flat-plate

### ~ thermal boundary layer ~

$$\Delta^2 f(\Delta) = \frac{39}{280}\Pr^{-1}$$

- case I: $\delta_T >> \delta$ $(\Pr << 1)$ i.e. $\Delta = \delta_T/\delta >> 1$

$$f(\Delta) = \left(-1 + 14\Delta^2 - 35\Delta^3 + 35\Delta^4\right)\frac{3}{280\Delta^4} \approx 35\cdot\frac{3}{280} = \frac{3}{8}$$

$$\Delta^2 = \frac{8}{3}\cdot\frac{39}{280}\Pr^{-1} = \frac{13}{35}\Pr^{-1} \Rightarrow \Delta = \sqrt{\frac{13}{35}}\Pr^{-1/2}$$

$$\delta_T(x) = 2.828\cdot x\cdot \mathrm{Re}_x^{-1/2}\cdot \Pr^{-1/2}$$

$$Nu = \frac{hx}{k} = 0.530\cdot \mathrm{Re}_x^{1/2}\cdot \Pr^{1/2}$$

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## Uniform flow over a flat-plate

### ~ thermal boundary layer ~

- case II: $\delta_T << \delta$ $(\Pr >> 1)$ i.e. $\Delta = \delta_T/\delta << 1$

$$\delta_T(x) = 4.53\cdot x\cdot \mathrm{Re}_x^{-1/2}\cdot \Pr^{-1/3}$$

$$Nu = \frac{hx}{k} = \frac{q_0''x}{k(T_0 - T_\infty)} = \frac{3x}{2\delta_T} = 0.331\mathrm{Re}_x^{1/2}\cdot \Pr^{1/3}$$

- case I: $\delta_T >> \delta$ $(\Pr << 1)$ i.e. $\Delta = \delta_T/\delta >> 1$

$$\delta_T(x) = 2.828\cdot x\cdot \mathrm{Re}_x^{-1/2}\cdot \Pr^{-1/2}$$

$$Nu = \frac{hx}{k} = 0.530\cdot \mathrm{Re}_x^{1/2}\cdot \Pr^{1/2}$$

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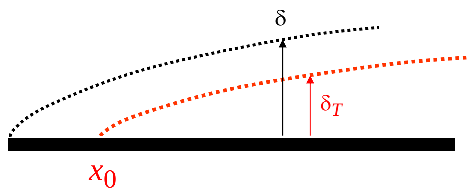
## Other Heating Conditions

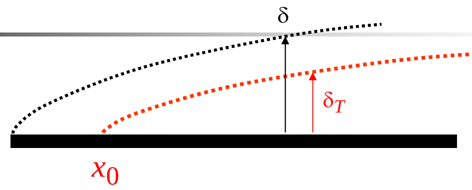
- non-uniform wall temperature $T_0 = T_0(x)$
- wall heat flux given $q_0''(x)$
- non-constant properties: use film temperature: $T_f \equiv \frac{1}{2}(T_0 + T_\infty)$
- flow over a body of arbitrary shape
- unheated starting length $T_0 = T_\infty$ for $0 \le x < x_0$ ; $T_0 \ne T_\infty$ for $x_0 \le x$






## Unheated Starting Length

~ flat-plate flow ~




Recall that for $\delta_T < \delta$ (case II):

Both velocity and temperature profiles are approximated as polynomials of degree 3.

$$\Delta_2 = \left(\frac{3}{20}\Delta - \frac{3}{280}\Delta^3\right)\cdot\delta_T \approx \frac{3}{20}\Delta\cdot\delta_T \quad \text{for } \delta_T << \delta$$

$$\delta_T \frac{d}{dx} f(\Delta)\delta_T = \frac{3}{2}\frac{\alpha}{U_\infty} \qquad I.C. \quad \delta_T(x_0) = 0$$

P.S. $\Delta = \frac{\delta_T}{\delta} = \Delta(x)$ now

## Unheated Starting Length

### ~ flat-plate flow ~

$$\delta_T \frac{d}{dx}\left(\frac{3\Delta}{20}\delta_T\right) = \frac{3}{2}\frac{\alpha}{U_\infty} \qquad I.C. \quad \delta_T(x_0) = 0$$

- write $\delta_T = \Delta\delta$
- use the result: $\delta(x) = \sqrt{\frac{280}{13}}\sqrt{\frac{\nu x}{U_\infty}} \quad \Rightarrow \quad \frac{d\delta}{dx} = \frac{1}{2}\sqrt{\frac{280}{13}\frac{\nu}{U_\infty x}} = \frac{\delta}{2x}$

$$\Rightarrow \quad \Delta\delta \frac{d}{dx}\left(\Delta^2\delta\right) = 10\frac{\alpha}{U_\infty}$$

$$\Rightarrow \quad \Delta\delta\left(\Delta^2 \cdot \frac{d\delta}{dx} + \delta \cdot 2\Delta \frac{d\Delta}{dx}\right) = \frac{10\alpha}{U_\infty}$$

$$\Rightarrow \quad \Delta^3 \cdot \frac{d\delta}{dx} + \delta \cdot 2\Delta^2 \frac{d\Delta}{dx} = \frac{10\alpha}{U_\infty \delta} \qquad \Rightarrow \quad \Delta^3 \cdot \frac{\delta}{2x} + \delta \cdot 2\Delta^2 \frac{d\Delta}{dx} = \frac{10\alpha}{U_\infty \delta}$$

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## Unheated Starting Length

### ~ flat-plate flow ~

$$\delta(x) = \sqrt{\frac{280}{13}}\sqrt{\frac{\nu x}{U_\infty}}$$

$$\Rightarrow \quad \Delta^3 \cdot \frac{\delta}{2x} + \delta \cdot 2\Delta^2 \frac{d\Delta}{dx} = \frac{10\alpha}{U_\infty \delta}$$

$$\Rightarrow \quad \Delta^3 + \frac{4x}{3}\frac{d\Delta^3}{dx} = \frac{20\alpha}{U_\infty}\frac{x}{\delta^2} = \frac{20\alpha}{U_\infty}\frac{13}{280}\frac{U_\infty}{\nu}$$

$$\Delta^3 + \frac{4}{3}x\frac{d\Delta^3}{dx} = \frac{13}{14}\mathrm{Pr}^{-1} \quad , \quad \Delta(x_0) = 0$$

$$\Rightarrow \quad \Psi + \frac{4x}{3}\frac{d\Psi}{dx} = \frac{13}{14}\mathrm{Pr}^{-1}, \qquad \Psi \equiv \Delta^3, \quad \Psi(x_0) = 0$$

$$\Psi(x) \neq \frac{13}{14}\mathrm{Pr}^{-1} \qquad \Rightarrow \quad \frac{4x}{3}\frac{d\Psi}{dx} = \frac{13}{14}\mathrm{Pr}^{-1} - \Psi \quad \Rightarrow \quad \frac{d\Psi}{\left(\frac{13}{14}\mathrm{Pr}^{-1} - \Psi\right)} = \frac{3}{4x}dx$$

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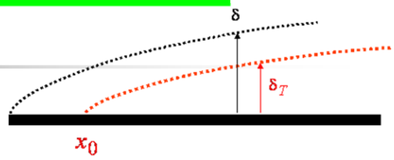
## *Unheated Starting Length*

*~ flat-plate flow ~*

$$\Rightarrow \int_0^{\Psi} \frac{d\Psi}{\left(\frac{13}{14}\mathrm{Pr}^{-1} - \Psi\right)} = \int_{x_0}^{x} \frac{3}{4x} dx$$




$$\Rightarrow -\ln\left| \frac{\left(\frac{13}{14}\mathrm{Pr}^{-1} - \Psi\right)}{\frac{13}{14}\mathrm{Pr}^{-1}} \right| = \frac{3}{4}\ln\left(\frac{x}{x_0}\right) \quad \Rightarrow \frac{\left|\frac{13}{14}\mathrm{Pr}^{-1} - \Psi\right|}{\frac{13}{14}\mathrm{Pr}^{-1}} = \left(\frac{x_0}{x}\right)^{3/4} \leq 1$$

$$\Rightarrow \frac{13}{14}\mathrm{Pr}^{-1} - \Psi = \pm\frac{13}{14}\mathrm{Pr}^{-1}\left(\frac{x_0}{x}\right)^{3/4} \quad \Rightarrow \Psi = \Delta^3 = \frac{13}{14}\mathrm{Pr}^{-1}\left\{1 \mp \left(\frac{x_0}{x}\right)^{3/4}\right\}$$

$$\Rightarrow \Delta = \frac{\delta_T}{\delta} = \left(\frac{13}{14}\right)^{1/3} \mathrm{Pr}^{-1/3} \cdot \left\{1 - \left(\frac{x_0}{x}\right)^{3/4}\right\}^{-1/3}$$

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## *Unheated Starting Length*

*~ flat-plate flow ~*

$$\delta(x) = \sqrt{\frac{280}{13}}\, x \cdot \mathrm{Re}_x^{-1/2}$$

$$\Delta = \frac{\delta_T}{\delta} = \left(\frac{13}{14}\right)^{1/3} \mathrm{Pr}^{-1/3} \cdot \left\{1 - \left(\frac{x_0}{x}\right)^{3/4}\right\}^{-1/3}$$

$$\Rightarrow \delta_T(x) = \left(\frac{13}{14}\right)^{1/3} \cdot \sqrt{\frac{280}{13}}\, x \cdot \mathrm{Re}_x^{-1/2} \cdot \mathrm{Pr}^{-1/3} \cdot \left\{1 - \left(\frac{x_0}{x}\right)^{3/4}\right\}^{-1/3}$$

$$\delta_T(x) \approx 4.528 \cdot \mathrm{Pr}^{-1/3} \cdot \mathrm{Re}_x^{-1/2} \cdot x \cdot \left\{1 - \left(\frac{x_0}{x}\right)^{3/4}\right\}^{1/3}$$

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## Unheated Starting Length

### ~ flat-plate flow ~

$$\delta_T(x) = 4.528 \cdot \Pr^{-1/3} \cdot \mathrm{Re}_x^{-1/2} \cdot x \cdot \left\{1 - \left(\frac{x_0}{x}\right)^{3/4}\right\}^{1/3}$$

$$q_0''(x) = \frac{3k(T_0 - T_\infty)}{2\delta_T} = \begin{cases} 0 & \text{for} \quad 0 \le x < x_0 \\ 0.332 \cdot \Pr^{1/3} \cdot \mathrm{Re}_x^{1/2} \cdot \dfrac{k(T_0 - T_\infty)}{x} \cdot \left\{1 - \left(\dfrac{x_0}{x}\right)^{3/4}\right\}^{-1/3} & \text{for} \quad x \ge x_0 \end{cases}$$

$$Nu = \frac{hx}{k} = \frac{q_0''(x)x}{k(T_0 - T_\infty)} = \begin{cases} 0 & \text{for} \quad 0 \le x < x_0 \\ 0.332 \cdot \Pr^{1/3} \cdot \mathrm{Re}_x^{1/2} \cdot \left\{1 - \left(\dfrac{x_0}{x}\right)^{3/4}\right\}^{-1/3} & \text{for} \quad x \ge x_0 \end{cases}$$

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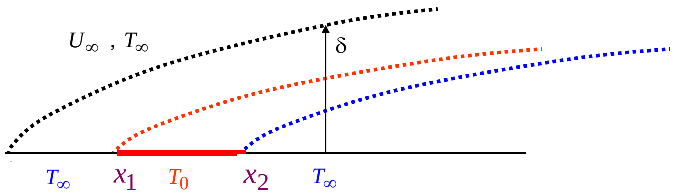
## Heated Spot

### ~ flat-plate flow ~




$$\frac{q_0''}{\rho_\infty U_\infty c_p (T_0 - T_\infty)} = \frac{d\Delta_2}{dx}$$

or $$-k\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{d}{dx}\int_0^{\delta_T} \rho c_p u (T - T_\infty) dy$$

or $$-k\left(\frac{\partial \hat{T}}{\partial y}\right)_{y=0} = \frac{d}{dx}\int_0^{\delta_T} \rho c_p u \hat{T} dy \qquad \text{where} \quad \hat{T} \equiv T - T_\infty$$

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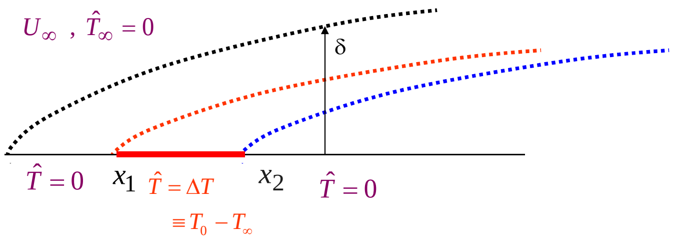
## Heated Spot

*~ flat-plate flow ~*




$$-k\left(\frac{\partial \hat{T}}{\partial y}\right)_{y=0} = \frac{d}{dx}\int_0^{\delta_T} \rho c_p u \hat{T} dy \quad \text{where} \quad \hat{T} \equiv T - T_\infty$$

⇨ The equation is linear in $\hat{T}$

- If $\hat{T}_1(x,y)$ and $\hat{T}_2(x,y)$ are both solutions, so is $\hat{T}_1(x,y)+\hat{T}_2(x,y)$
- If $\hat{T}_1(x,y)$ is a solution, so is $\sigma\hat{T}_1(x,y)$ , $\sigma \in R$

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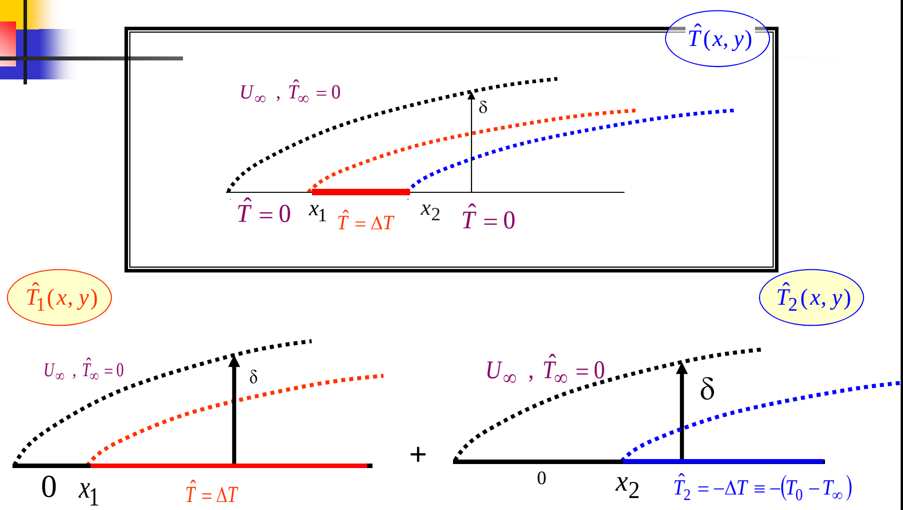
## Heated Spot

*~ flat-plate flow ~*

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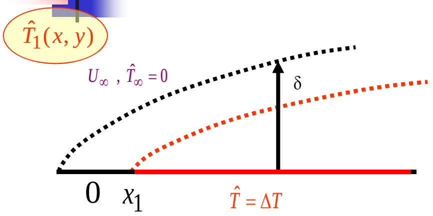
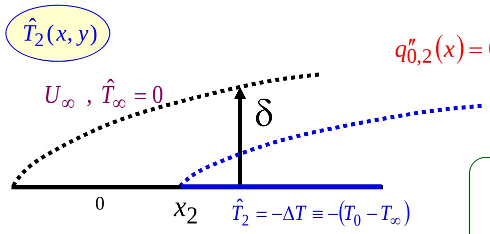
## Heated Spot
### ~ flat-plate flow ~




$$q''_{0,1}(x) = 0.332 \cdot \Pr^{1/3} \cdot \mathrm{Re}_x^{1/2} \cdot \frac{k\Delta T}{x} \cdot \left\{1 - \left(\frac{x_1}{x}\right)^{3/4}\right\}^{-1/3}$$

$x_1 \le x$




$$q''_{0,2}(x) = 0.332 \cdot \Pr^{1/3} \cdot \mathrm{Re}_x^{1/2} \cdot \frac{-k\Delta T}{x} \cdot \left\{1 - \left(\frac{x_2}{x}\right)^{3/4}\right\}^{-1/3}$$

$x_2 \le x$

$$q''_0(x) = -k\left(\frac{\partial \hat{T}}{\partial y}\right)_{y=0} = q''_{0,1}(x) + q''_{0,2}(x)$$

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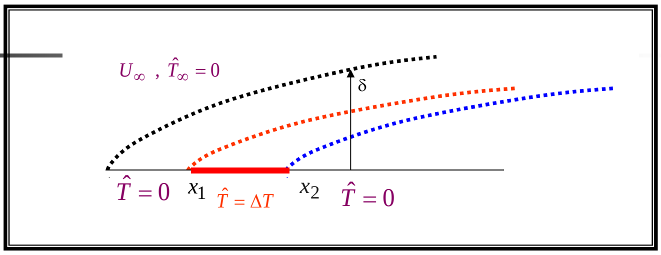
## Heated Spot
### ~ flat-plate flow ~




$$q''_0(x) = \begin{cases} 0 & \text{for } 0 \le x < x_1 \\ 0.332 \cdot \Pr^{1/3} \cdot \mathrm{Re}_x^{1/2} \cdot \frac{k\Delta T}{x} \cdot \left[1 - \left(\frac{x_1}{x}\right)^{3/4}\right]^{-1/3} & \text{for } x_1 \le x < x_2 \\ 0.332 \cdot \Pr^{1/3} \cdot \mathrm{Re}_x^{1/2} \cdot \frac{k\Delta T}{x} \cdot \left\{\left[1 - \left(\frac{x_1}{x}\right)^{3/4}\right]^{-1/3} - \left[1 - \left(\frac{x_2}{x}\right)^{3/4}\right]^{-1/3}\right\} & \text{for } x_2 \le x \end{cases}$$

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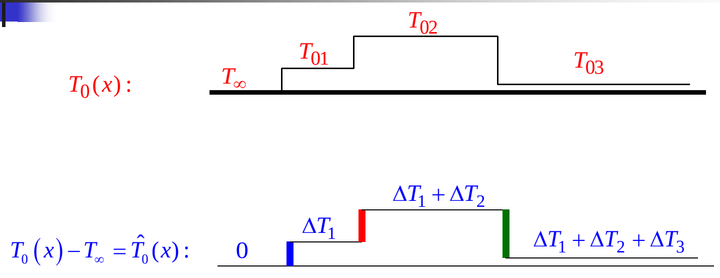
## Heated Spots

*~ flat-plate flow ~*




$$\hat{T} = T - T_\infty$$

$$\Delta T_i \equiv T_{0,i+1} - T_{0,i}$$

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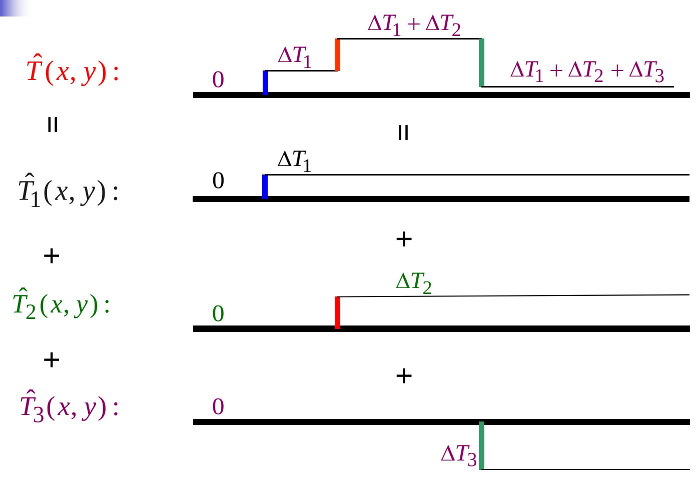
## Heated Spots

*~ flat-plate flow ~*

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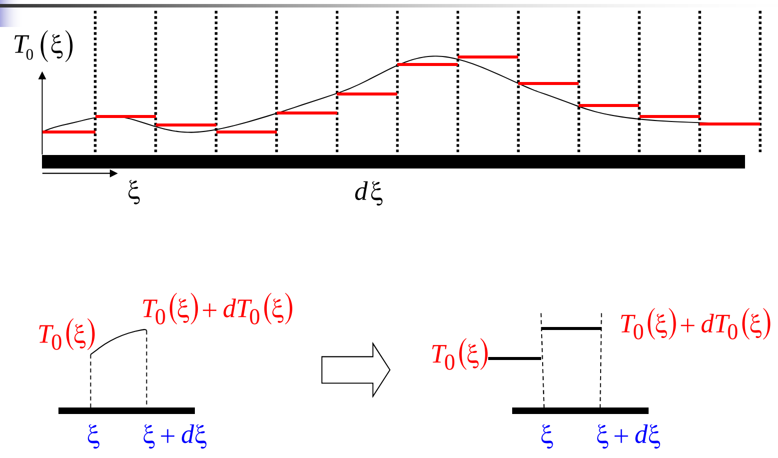
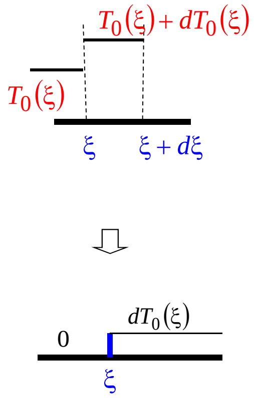
## Arbitrary wall temperature
### ~ flat-plate flow ~






## Arbitrary wall temperature
### ~ flat-plate flow ~

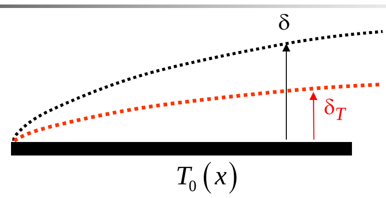



$$q''_{0,\xi}(x) = \begin{cases} 0 & \text{for} \quad 0 \le x < \xi \\ 0.332 \cdot \Pr^{1/3} \cdot \mathrm{Re}_x^{1/2} \cdot \dfrac{k\,dT_0(\xi)}{x} \cdot \left\{1-\left(\dfrac{\xi}{x}\right)^{3/4}\right\}^{-1/3} & \text{for} \quad \xi \le x \end{cases}$$

$$q''_0(x) = \sum_{\xi \le x} q''_{0,\xi}(x)$$

$$q''_0(x) = \int_0^x 0.332 \cdot \Pr^{1/3} \cdot \mathrm{Re}_x^{1/2} \cdot \frac{k}{x} \cdot \left\{1-\left(\frac{\xi}{x}\right)^{3/4}\right\}^{-1/3} \frac{dT_0(\xi)}{d\xi} d\xi$$

## Wall Heat Flux
### ~ flat-plate flow ~




Expect: $T_0(0) = T_\infty$ and $T_0(x) \uparrow$ as $x \uparrow$ for $q_0''(x) > 0$

Also expect: $\delta_T < \delta$ because initially the temperature difference is small.

Polynomial of 3rd degree: $\Delta_2 \approx \frac{3\Delta}{20}\delta_T$



## Wall Heat Flux
### ~ flat-plate flow ~

$$\frac{q_0''}{\rho c_p U_\infty} = \frac{d}{dx}\left\{\Delta_2\left(T_0 - T_\infty\right)\right\} \qquad \delta(x) = \sqrt{\frac{280}{13}} \cdot x \cdot \mathrm{Re}_x^{-1/2}$$

Recall (polynomial of 3rd degree) : $q_0''(x) = \frac{3}{2}\frac{k\Delta T(x)}{\delta_T(x)}$, $\quad \Delta T(x) \equiv T_0(x) - T_\infty$

$$\Delta_2 \approx \frac{3\Delta}{20}\delta_T$$

$$\frac{\int_0^x q_0''(\xi)\,d\xi}{\rho c_p U_\infty} = \Delta_2 \cdot \Delta T\Big|_0^x = \Delta_2(x)\cdot\Delta T(x)$$

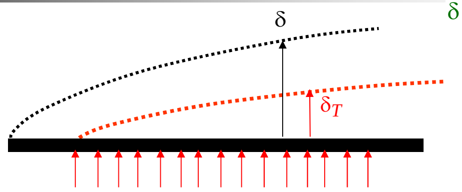
$$= \frac{3\Delta}{20}\cdot\delta_T\cdot\Delta T = \frac{3\delta_T}{20\delta}\cdot\delta_T\cdot\Delta T \;= \frac{3}{20\delta}\cdot\left(\frac{3}{2}\frac{k\Delta T}{q_0''}\right)^2\cdot\Delta T = \frac{27}{80}\cdot\frac{k^2}{q_0''^2}\cdot\frac{1}{\delta}\cdot\Delta T^3$$

$$\Delta T^3(x) = \frac{80\delta}{27}\frac{q_0''^2(x)}{k^2}\frac{\int_0^x q_0''(\xi)\,d\xi}{\rho c_p U_\infty}$$

## Wall Heat Flux
### ~ flat-plate flow ~

if $q_0'' = 0$ for $0 \le x < x_0$ and $q_0'' = \text{constant}$ for $x \ge x_0$

$$\delta(x) = \sqrt{\frac{280}{13}} x \cdot \text{Re}_x^{-1/2}$$




$$\Delta T^3(x) = \frac{80\delta}{27} \frac{q_0''^2(x)}{k^2} \frac{\int_0^x q_0''(\xi)\, d\xi}{\rho c_p U_\infty} = \begin{cases} 0 & \text{for } 0 \le x < x_0 \\ \dfrac{80\delta}{27} \dfrac{q_0''^2}{k^2} \dfrac{q_0'' \cdot (x - x_0)}{\rho c_p U_\infty} & \text{for } x_0 \le x \end{cases}$$

$$\Delta T(x) = T_0(x) - T_\infty = \begin{cases} 0 & \text{for } 0 \le x < x_0 \\ q_0'' \left( \dfrac{80\delta}{27k^2} \dfrac{(x - x_0)}{\rho c_p U_\infty} \right)^{1/3} & \text{for } x_0 \le x \end{cases}$$



## Wall Heat Flux
### ~ flat-plate flow ~

if $q_0'' = 0$ for $0 \le x < x_0$ and $q_0'' = \text{constant}$ for $x \ge x_0$

$$\Delta T(x) = T_0(x) - T_\infty = \begin{cases} 0 & \text{for } 0 \le x < x_0 \\ 2.396 \cdot \dfrac{q_0''}{k} \left( x \cdot \text{Re}_x^{-1/2} \dfrac{\alpha (x - x_0)}{U_\infty} \right)^{1/3} & \text{for } x_0 \le x \end{cases}$$

$$Nu = \frac{h(x) x}{k} = \frac{q_0''(x) \cdot x}{k \{ T_0(x) - T_\infty \}} = \begin{cases} 0 & \text{for } 0 \le x < x_0 \\ 0.417 \cdot \text{Re}_x^{1/2} \cdot \text{Pr}^{1/3} \cdot \left( 1 - \dfrac{x_0}{x} \right)^{1/3} & \text{for } x \ge x_0 \end{cases}$$